

3rd Annual Shepherd Open Mathematics Contest, 2008

- I** A point P is moving on side AC of an acute angled triangle ABC. We construct a rectangle inside the triangle with one vertex at P, the others on the sides of the triangle, with two vertices on side BC. Denoting the center of this rectangle by O, find the locus of O !
(Locus means the set of all points where the point O can be located as P moves on AC. *Hint*: experiment ! Draw several pictures, try to first guess the answer, then prove that your answer is correct.)
- II** Two people are playing the following game, using a chessboard: the first can move a piece from the lower left corner horizontally to the right to a square of his choice, the second player moves the piece vertically upwards (from the square on which the first player left it) to a square of his choice. They keep taking turns this way, the first player always moving to the right, the second player upwards. The player who first reaches the upper right corner, wins.
Does either player have a winning strategy ? If yes, what is this strategy, if not, why not ?
- III** Evaluate the expression E below in terms of a, b and c :

$$E = \left(\frac{x-1}{x+1} \right)^2 + \left(\frac{y-1}{y+1} \right)^2 + \left(\frac{z-1}{z+1} \right)^2$$

where

$$x = \frac{a \pm \sqrt{a^2 - (b+c)^2}}{b+c}, \quad y = \frac{b \pm \sqrt{b^2 - (c+a)^2}}{c+a}, \quad z = \frac{c \pm \sqrt{c^2 - (a+b)^2}}{a+b}$$

(assuming of course that no denominators are zero).

(*Hint*: you can try the brute force method, but the expressions above for x, y and z are suggestive of the quadratic formula. Try thinking of x, y, z as roots of certain quadratic equations. Your final answer will be particularly "nice".)